A topological game on the space of ultrafilters

José Carlos Fontanesi Kling

jose.kling@pm.me

Instituto de Ciências Matemáticas e de Computação
ICMC - USP

Supervisor: Prof. Dr. Leandro Fiorini Aurichi

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- A play is an infinite string of pairwise distinct natural numbers $(a_1, b_1, a_2, b_2, \dots)$, the terms a_n indicating *Alice*'s choices and b_n *Bob*'s choices.

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- A play is an infinite string of pairwise distinct natural numbers $(a_1, b_1, a_2, b_2, \dots)$, the terms a_n indicating *Alice*'s choices and b_n *Bob*'s choices.
- *Alice* wins if the set of her choices during the game is in T, that is, $\{a_1, a_2, a_3, \dots\} \in T$. *Bob* wins otherwise.

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- *Alice* wins if the play is one of the following sets:

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We are most interested in some specific targets, namely ultrafilters and sets that arise in Ramsey theoretical results, such as IP-sets and AP-rich sets.

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- They cannot repeat previous choices.
- *Alice* wins if $\overline{\{a_1, a_2, \dots\}} \cap T \neq \emptyset$.

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- If $A \in \mathcal{S}$ and $A \subset B$, then $B \in \mathcal{S}$.
- If $A \cup B \in \mathcal{S}$, then $A \in \mathcal{S}$ or $B \in \mathcal{S}$.

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- *Alice* wins $F(T)$ for $T \subset \omega^*$ open or dense.

Relations with other games

Let $\mathcal{F}$ be a filter. In the game $\mathcal{G}(\mathcal{F})$ *Alice* and *Bob* take turns choosing a natural number (may be repeated). *Bob* wins if his choices eventually dominates *Alice*'s choices and the set of his choices is in $\mathcal{F}$.

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Theorem (Bartoszyński and Scheepers)

Alice has a winning strategy in $\mathcal{G}(\mathcal{F})$ if, and only if, $\mathcal{F}$ is not a rare filter. (A rare ultrafilter is called a q-point)

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As a consequence, we get that if $p \in \omega^*$ is a q-point, then none of the players have a winning strategy in $F(p)$.

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Let X be a complete metric space and $T \subset X$, then

- *Alice has winning strategy if in $BM(X,T)$, and only if, T is comeager in some open set of X.*
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An ultrafilter $p \in \omega^*$ is not meager nor comeager in 2^ω, so neither player has a winning strategy in $F_{fin}(p)$.

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Theorem (Talagrand)

A superfilter $\mathcal{S} \subset 2^\omega$ is comeager if, and only if, there is a partition $I_1, I_2, \ldots$ of ω in finite intervals such that for all infinite $N \subset \omega$, $\bigcup_{n \in N} I_n \in \mathcal{S}$. (Thanks to Andreas Blass)

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As a consequence we get that if $T \subset 2^\omega$ is the union of countable ultrafilters, then it is not comeager.
Corolary: If $T \subset \omega^*$ is a countable set, then none of the players have a winning strategy in $F_{fin}(T)$.

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- Does *Alice* have a winning strategy in $F_{fin}(T)$ if $T \subset \omega^*$ is an uncountable set?